Math 153 Fall 2009 - Problem Set #6: Games on Graphs

The discussion on p. 117-118 and p. 139-142 of the text may be useful for this problem set. Do tell us if you have questions about that material. Please show all work and represent your answers in simplest possible form. Submit your code for Question #2 along with your answers.

Question #1 (50 points)

Consider a game between two strategies, A and B, given by the payoff matrix

$$\begin{array}{ccc} & A & B \\ A & a & b \\ B & c & d \end{array}$$

The population of size N is arranged on a cycle. Each player interacts with its two immediate neighbors. The payoffs from these two interactions are added up. The fitness of an individual is given by $(1 - w + wP)$ where P is the individual's payoff and w is the intensity of selection. We consider the Birth-Death (BD) update rule: an individual is selected for reproduction proportional to fitness and the offspring replaces a randomly chosen neighbor. We start with a population of all B and introduce one A mutant. We want to find the fixation probability of this A mutant.

(a) (5 points) Describe the states of the system. How many states are there?

(b) (15 points)
What are the transition probabilities between the states? How do they look in the limit of weak selection? How does the stochastic matrix for the process look?

(c) (15 points)
What is the ratio of the fixation probabilities, ρ_A/ρ_B in the limit of weak selection?

(Hint: Use Taylor expansions. What is $\frac{1+wx}{1+wy}$ in the limit of small w?)

(d) (15 points)
What is the condition for A to be risk-dominant over B? How does this condition relate to the ratio of fixation probabilities?

Question #2 (50 points) Consider the prisoner's dilemma:

$$\begin{array}{ccc} & C & D \\ C & b-c & -c \\ D & b & 0 \end{array}$$

Normalize the cost parameter c to equal 1. The benefit b may vary.

The population of size N is arranged on a cycle. Each player interacts with its two immediate neighbors. The payoffs from these two interactions are added up. The fitness of an individual is given by $(3 - w + wP)$ where P is the individual's payoff and w is the intensity of selection (between 0 and 1, inclusive).

Note that the minimum possible fitness is zero, obtained for the sucker's payoff and $w = 1$.

There is no mutation. We consider two different update rules:

- Birth-Death (BD): First an individual is selected for reproduction proportional to fitness. Its offspring replaces a randomly chosen neighbor.
- Death-Birth (DB): First an individual is selected for death, at random, without regard to fitness. One of its neighbors is chosen, proportional to fitness, to replace it.

Write a program in Python (or any other language) to simulate both processes. Submit the code with your answers. Your program will need to:

- Keep track of the type ($0 = A$, $1 = B$) of each of the N players, using a list of length N.
- Let player i interact with player $i-1$ and $i+1$ for $1 < i < N$. Let player 1 interact with player 2 and player N; let player N interact with player $N-1$ and player 1.
- Each generation, you will:
  - Calculate the payoff of each player, given that they play one game with each neighbor.
  - Calculate the fitness of each player, using the w parameter.
  - Perform the update procedure, based on fitness, as described above.
- Repeat this loop until every player in the population has the same strategy, at which point the simulation ends.

(a) In your simulations, use a population size of $N = 20$. Run the following 72 scenarios (42 points):

- 2 update rules (BD & DB)
- For each, try 6 values for b: 1, 2, 3, 4, 5, 100
- And for each, try 6 values for w: 0, 0.05, 0.1, 0.3, 0.9, 1.0

For each scenario, do at least **5,000 runs**. Start each run with a population of 19 defectors and a single cooperator.

Compute the following:

- Probability that cooperation goes to fixation
- 95% confidence interval around this probability (defined as $\rho \pm 1.96\sqrt{\frac{\rho(1-\rho)}{r}}$), where ρ is your computed fixation probability and r is the number of runs (at least 5,000).

Graph your results. Plot fixation probability ρ against w, and show the 95% CI bars. You should produce 12 separate lines (2 update rules * 6 values for b).

(FYI: Each scenario with 5,000 runs took me about 60 seconds for my laptop to complete, so set aside enough time to run all 72 of them.)

(b) Now consider only the case of weak selection (approximated using $w = 0.05$) (8 points). Plot fixation probability ρ against b, and show the 95% CI bars. For comparison, plot a horizontal line representing the fixation probability of a neutral mutant.

- For which values of b does evolution favor the fixation of cooperation?
- For which values of b does evolution oppose the fixation of cooperation?
- And for which values of b is the fixation probability of cooperation not significantly different from the neutral probability of fixation (using 95% confidence intervals)?