Virus Dynamics & Replicator Equation Problem Set

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Please show all work and represent your answers in simplest possible form.

Question #1 (50 points)

1a. (10 points) Consider the following model of HIV infection:

- In the absence of virus particles, the number of uninfected cells, x, is governed by the equation $\dot{x} = \lambda - dx$.
- Infected cells produce free virus at rate k. Free virus dies at rate u.
- Infected cells are produced when free virus infects uninfected cells at rate β.
- Infected cells die at rate a.

Let v denote the number of free virus particles, and y the number of infected cells. Write down the system of differential equations that desribes the above dynamics of x, y and v.

1b. (40 points) Here we introduce drug treatment to the above system.

(i) Rewrite the differential equations from (1a), now assuming that treatment prevents the free virus from infecting uninfected cells.

(ii) Using the system that you wrote down in (i), find the general solution for $y(t)$ and $v(t)$. (You do not need to assume any particular initial conditions.)

(iii) Let v^* and y^* be the (nonzero) equilibrium values of virus load and infected cells, respectively, prior to treatment. Assume that at the start of treatment, the virus load is equal to v^* and the number of infected cells is equal to y^*. Using your result from part (ii), write down the specific solution for $v(t)$, in terms of v^*, u, and a.

(iv) Plot $v(t)$ on a logarithmic scale (*i.e.*, plot $\log(v(t))$, as a function of t). Use parameter values $u = 10$, $a = 0.1$, and $v^* = 100$.

(v) Assuming that a is much smaller than u, how can we estimate the average lifetime of an infected cell by using the plot in (iv)?

Question #2 (50 points)

This question explores the replicator equation, given on p. 56 of the text.

2a. (25 points)

Consider the 2x2 payoff matrix, with strategies **A** and **B**:

	A	**B**
A	a	b
B	c	d

(i) (5 pts) Write the formula for $\frac{dx}{dt}$ (in terms of x and the payoff values) that describes the replicator dynamics for this 2-strategy game.

(ii) (10 pts) A strategy is called an *evolutionarily stable strategy (ESS)* if the population consisting entirely of that strategy is a stable equilibrium. Derive the condition (in terms of a, b, c, d) that determines whether strategy **A** is an ESS. Do the same for strategy **B**.

(iii) (5 pts) We have constant selection if $a = b$ and $c = d$. *Fisher's Fundamental Theorem* states that, in cases of constant selection, the rate at which average fitness increases is equal to the population's variance of fitness. Prove the theorem for this 2-strategy case.

(iv) (5 pts) Give an example scenario where average fitness decreases over time. (Choose numbers for a, b, c, d, and for the initial conditions, such that average fitness decreases. Calculate both initial average fitness and equilibrium average fitness.)

2b. (25 points)

Consider an n-by-n payoff matrix, P, with strategies $S_1 \ldots S_n$, where x_i is the frequency of strategy S_i in the population. The payoff for strategy S_i versus S_j is given by P_{ij}.

Definitions:

- Strategy S_i is said to *strictly dominate* strategy S_j if, for all k, $P_{ik} > P_{jk}$. (Intuitively, S_i is always a "better choice" than S_j, no matter what one's opponent does.)
- A strategy is *strictly dominated* if there exists another strategy that strictly dominates it.
- $f_i(\vec{x})$ is the current fitness of strategy S_i, given the current population vector $\vec{x}$.

(i) (10 pts) Show the following, assuming that $x_j > 0$:

$$\frac{d\left(\frac{x_i}{x_j}\right)}{dt} = \left(\frac{x_i}{x_j}\right)\left(f_i(\vec{x}) - f_j(\vec{x})\right)$$

(ii) (10 pts) Suppose that strategy S_d is strictly dominated. Prove that, for any initial condition with all positive frequencies (that is, $\forall i\,(x_i > 0)$), the frequency x_d limits to zero as $t \to \infty$. (You can use the fact that, if all strategy frequencies start out positive, they remain positive for all time.)

Selection against iteratively strictly dominated strategies:

More Definitions:

- One can define a new payoff matrix, $P^{(2)}$, which is constructed from P by eliminating all rows and columns corresponding to strictly dominated strategies of P. In this new game, there may be other strictly dominated strategies.
- One can iterate the process, defining payoff matrix $P^{(m+1)}$ by eliminating all rows and columns corresponding to the strictly dominated strategies of $P^{(m)}$.
- A strategy that is eliminated as a result of this iterative procedure is called an *iteratively strictly dominated strategy.*

(iii) (5 pts) Prove the result in (ii), but now assume that S_d is just iteratively strictly dominated.